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Elliptic systems with nonlinearity q greater or equal than two with controlled growth. Global Hölder continuity of solutions to the Dirichlet problem

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Abstract

Hölder regularity up to the boundary of the solutions to the Dirichlet problem for second order elliptic systems with nonlinearity $q > 2$ and with controlled growth is proved when $n \leqslant q + 2$.

Keywords: Nonlinear elliptic systems; Global Hölder regularity; Higher gradient summability

1. Introduction

The aim of this paper is to study the global Hölder continuity in $\bar{\Omega}$ of a solution $u \in H^{1,q}(\Omega)$ to the following Dirichlet problem

$$\begin{cases} u - g \in H_0^{1,q}(\Omega), \\ \sum_{i=1}^n D_i a^i(x, Du) = \sum_{i=1}^n D_i F^i(x, u) - F^0(x, u, Du) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $n \geqslant 2$ and q is a real number $\geqslant 2$.

For a solution u to (1.1) we mean that $u = g + w$, where $w \in H_0^{1,q}(\Omega)$ is such that

$$\int_\Omega \sum_{i=1}^n \big(a^i(x, Dw + Dg)|D_i\varphi\big)\, dx = \int_\Omega \sum_{i=1}^n \big(F^i(x, w + g)|D_i\varphi\big)\, dx$$

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$$+\int_\Omega \bigl(F^0(x, w+g, Dw+Dg)|\varphi\bigr)\,dx, \quad \forall\varphi \in H_0^{1,q}(\Omega). \tag{1.2}$$

We denote by u a vector $\Omega \to \mathbb{R}^N$, $N > 1$, and $Du = (D_1u, D_2u, \dots, D_nu)$. If $u, v \in \mathbb{R}^N$, $(u|v)$ denotes the inner product in $\mathbb{R}^N$. We set $p = (p^1, \dots, p^n)$, with $p^i \in \mathbb{R}^N$; p is a typical vector of $\mathbb{R}^{nN}$.

For every $p \in \mathbb{R}^K$, $K \geqslant 1$, we set

$$V(p) = \bigl(1 + \|p\|^2\bigr)^{\frac{1}{2}} \quad \text{and} \quad W(p) = V^{\frac{q-2}{2}}(p)p. \tag{1.3}$$

Let $a^i(x, p)$, $i = 1, 2, \dots, n$, be vectors of $\mathbb{R}^N$, defined on $\Omega \times \mathbb{R}^{nN}$, of class C^1 in p and uniformly continuous in x in the following sense: for every $x, y \in \Omega$ and $p \in \mathbb{R}^{nN}$

$$\Bigl\{\sum_i \bigl\|a^i(x, p) - a^i(y, p)\bigr\|^2\Bigr\}^{\frac{1}{2}} \leqslant \omega\bigl(\|x - y\|\bigr)V^{q-2}(p)\|p\|, \tag{1.4}$$

where $\omega(t)$, with $t > 0$, is a bounded, nondecreasing function, converging to zero as $t \to 0$. We suppose that

$$\begin{cases} a^i(x, 0) = 0 \quad \forall x \in \Omega, \\ \frac{\partial a^i}{\partial p_k^j} \text{ are measurable in } x \text{ and continuous in } p. \end{cases} \tag{1.5}$$

Setting

$$A_{ij}^{hk}(x, p) = \frac{\partial a_h^i(x, p)}{\partial p_k^j}, \qquad A_{ij} = \bigl\{A_{ij}^{hk}\bigr\}, \tag{1.6}$$

$$\tilde{A}_{ij}(x, p) = \int_0^1 A_{ij}(x, tp)\,dt, \tag{1.7}$$

we suppose that, $\forall x \in \Omega$, $\forall p \in \mathbb{R}^{nN}$ and $\forall \xi \in \mathbb{R}^{nN}$,

$$\Bigl\{\sum_{ij} \bigl\|A_{ij}(x, p)\bigr\|^2\Bigr\}^{\frac{1}{2}} \leqslant MV^{q-2}(p), \tag{1.8}$$

$$\sum_{ij}\bigl(A_{ij}(x, p)\xi^j|\xi^i\bigr) \geqslant \nu V^{q-2}(p)\|\xi\|^2, \tag{1.9}$$

where M and ν are positive constants.

In virtue of hypothesis (1.5)

$$a^i(x, p) = \sum_j \tilde{A}_{ij}(x, p)p^j,$$

and by condition (1.8)

$$\bigl\|a^i(x, p)\bigr\| \leqslant MV^{q-2}(p)\|p\|. \tag{1.10}$$

Moreover, let us denote by q^* the number $nq/(n - q)$ if $q < n$ and any number greater or equal than n if $q = n$, by q' and q'' the numbers such that $1/q + 1/q' = 1$ and

$1/q^* + 1/q'' = 1$, respectively. To assure the existence of the integrals which appear in (1.1), we assume that $F^0(x,u,p)$ and $F^i(x,u)$, $i = 1,\dots,n$, are vectors of $\mathbb{R}^N$, defined, respectively, in $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ and in $\Omega \times \mathbb{R}^N$, measurable in x, continuous in u and p, such that

$$\begin{aligned} &\|F^0(x,u,p)\| \leqslant C_0\big(1 + \big(\|u\|^\beta + \|p\|^\gamma\big)^{q-1}\big),\\ &\|F^i(x,u)\| \leqslant C_1\big(1 + \|u\|^{\alpha(q-1)}\big), \quad i = 1,2,\dots,n, \end{aligned} \tag{1.11}$$

where

$$1 \leqslant \alpha \leqslant \frac{q^*}{q}, \qquad 1 \leqslant \beta \leqslant \frac{q^*-1}{q-1}, \qquad 1 \leqslant \gamma \leqslant \frac{q'}{q''} \tag{1.12}$$

and C_0 and C_1 are positive constants.

Conditions (1.11), (1.12) are called controlled growth conditions and the aim of this paper is to study the global Hölder continuity in $\bar{\Omega}$ of the solutions to (1.1) under these conditions.

It is well known that it is not possible to achieve the global Hölder continuity in $\bar{\Omega}$ for each value of the dimension, as the examples given in [5] and [6] show. For $n < q$ the desired regularity derives from the Sobolev imbedding theorems. If $n = q$ we get the global Hölder continuity taking into account the result of higher summability of Du given by Lemma 3.3, which ensures that, if $\partial\Omega$ is of class C^2 and $g \in H^{1,q^*}(\Omega)$, there exists a number $r > 1$ such that $u \in H^{1,qr}(\Omega)$. On the other hand, if $n > q$ the main result of this paper ensures the desired regularity surely for $n \leqslant q + 2$, provided that g belongs to the space

$$H^{1,\frac{nq}{n-q},(\frac{\mu}{q-1})}(\Omega) = \big\{g \in H^{1,\frac{nq}{n-q}}(\Omega)\colon\ Dg \in L^{\frac{nq}{n-q},\frac{\mu}{q-1}}(\Omega)\big\}$$

(see Section 2 for the notations). Indeed, we show the following theorem:

Theorem 1.1. *Assume that conditions* (1.4), (1.5), (1.8), (1.9), (1.11) *and* (1.12)[2] *are fulfilled. Let $\partial\Omega$ be of class C^2, and*

$$g \in H^{1,\frac{nq}{n-q},(\frac{\mu}{q-1})}(\Omega), \quad 0 < \frac{\mu}{q-1} < \lambda.$$

[2] The result holds also true if instead of (1.11) one considers the conditions

$$\begin{aligned} &\|F^0(x,u,p)\| \leqslant |f_0(x)| + \big(|b(x)|\|u\|^\beta + |c(x)|\|p\|^\gamma\big)^{q-1},\\ &\|F^i(x,u)\| \leqslant |f_i(x)| + |a(x)|\|u\|^{\alpha(q-1)}, \quad i = 1,2,\dots,n, \end{aligned}$$

with $a(x), b(x), c(x) \in L^\infty(\Omega)$,

$$\begin{aligned} &f_i(x) \in L^{\frac{q}{q-1},\frac{\mu}{q-1}}(\Omega), \quad i = 1,\dots,n, \quad 0 < \frac{\mu}{q-1} < \lambda,\\ &f_0(x) \in L^{qs,\mu s}(\Omega), \end{aligned}$$

where, for $n > q$, $s = n/(n(q-1)+q)$ and, for $n = q$, q is a number $\in (1,n)$ (see [10] for details). For the notations see Section 2.

Then there exists a number $\lambda = \lambda(\nu, M, n)$, $2 < \lambda \leqslant n$, *such that if* $u \in H^{1,q}(\Omega)$ *is a solution of the Dirichlet problem* (1.1), *we get*

$$Du \in L^{q,\frac{\mu}{q-1}}(\Omega) \tag{1.13}$$

and, if $n - q < \mu/(q-1) < \lambda$,

$$u \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^N) \quad \textit{with } \alpha = 1 - \frac{n}{q} + \frac{\mu}{q-1}\frac{1}{q}.$$

To obtain the result of Theorem 1.1, we follow in outline a technique due to Campanato, which requests to get higher regularity of Du in a suitable Morrey space $L^{q,\lambda}(\Omega)$ and then the regularity of u in the Campanato space $\mathcal{L}^{q,\lambda+q}(\Omega)$; therefore we obtain the Hölder continuity of u, in virtue of an isomorphism property for suitable range of parameters between $\mathcal{L}^{q,\lambda+q}(\Omega)$ and $C^{0,\alpha}(\bar{\Omega})$. In paper [3] the global Hölder continuity is studied for a Dirichlet problem with the terms F^i which do not depend on u and Du; in paper [4] only the interior regularity for solutions to nonlinear systems with controlled growth is considered, whereas in [10] the author is concerned with the case of inhomogeneity with linear growth.

An essential tool, which has interest in itself, in order to obtain our regularity results is the global higher summability of Du, which we can deduce by showing the so-called Caccioppoli type inequality, both in the interior case and near the boundary.

It is worth remarking that if the vector a^i depend also on u, the above global Hölder continuity result for $q \leqslant n \leqslant q + 2$ does not hold, as the example in [7] shows.

In the general case the result we can expect if $q > n$ is only the so-called "partial Hölder regularity," namely there exists a closed singular set Ω_0 such that u is Hölder continuous in $\Omega \setminus \Omega_0$ (see [4,7,14]). This behaviour seems to be also true if the nonhomogeneous terms have natural growth in p, namely with growth of the type p^q, provided that a smallness condition in $\|u\|_{L^\infty(\Omega)}$ is verified (see [8,14] for the case $q = 2$). Moreover, also in the case of elliptic nonvariational system the global Hölder regularity up to the boundary is obtained for low values of n, namely $n < 6$ (see [12]).

We recall also that regularity results for elliptic systems with arbitrary order equations have been considered by Widman (see [13]), who establishes, under less restrictive assumptions, the Hölder continuity of solutions if $n < q + \varepsilon$, with $\varepsilon > 0$.

Finally we mention that the results of this paper have been presented at the First AMS-UMI Joint Meeting, Pisa, June 2002, and an abridged version of this paper can be found in [11].

2. Preliminary results

We define

$$B(x^0, \sigma) = \{x\colon \|x - x^0\| < \sigma\}; \tag{2.1}$$

moreover, if $x^0_n = 0$,

$$B^{+}(x^{0},\sigma)=\{x\in B(x^{0},\sigma):\ x_{n}>0\},\tag{2.2}$$

$$\Gamma(x^{0},\sigma)=\{x\in B(x^{0},\sigma):\ x_{n}=0\}.\tag{2.3}$$

We will simply write $B^{+}(\sigma)$, $\Gamma(\sigma)$ and Γ instead of $B^{+}(0,\sigma)$, $\Gamma(0,\sigma)$ and $\Gamma(0,1)$, respectively.

Through the present paper, Ω will denote a bounded open set of $\mathbb{R}^{n}$ with diameter d_{Ω} and with $\partial\Omega$ of class C^{2}.

The notation $B(x^{0},\sigma)\subset\subset\Omega$ means that $\overline{B(x^{0},\sigma)}\subset\Omega$.

Moreover, if $u\in L^{1}(\mathcal{B})$ and $\mathcal{B}$ is a measurable set with $\operatorname{meas}\mathcal{B}\neq 0$, then

$$u_{\mathcal{B}}=\fint_{\mathcal{B}}u(x)\,dx=\frac{1}{\operatorname{meas}\mathcal{B}}\int_{\mathcal{B}}u(x)\,dx.\tag{2.4}$$

If $u\in L^{\infty}(\Omega)$, we define

$$\|u\|_{\infty,\Omega}=\operatorname*{ess\,sup}_{\Omega}\|u(x)\|.\tag{2.5}$$

If $u\in C^{0,\alpha}(\bar{\Omega})$, $0<\alpha\leqslant 1$, we set

$$[u]_{\alpha,\bar{\Omega}}=\sup_{x,y\in\Omega}\frac{\|u(x)-u(y)\|}{\|x-y\|^{\alpha}}\tag{2.6}$$

and we will say that $u\in C^{0,\alpha}(\Omega)$ if $u\in C^{0,\alpha}(K)$ for every compact subset $K\subset\Omega$.

Let us recall the definition of the spaces $L^{q,\mu}(\Omega)$ and $\mathcal{L}^{q,\mu}(\Omega)$ (for more details see [1] and [2]).

$L^{q,\mu}(\Omega)$, $0\leqslant\mu\leqslant n$, $q\geqslant 1$, is the space of those functions $u\in L^{q}(\Omega)$ such that

$$\|u\|^{q}_{L^{q,\mu}(\Omega)}=\sup_{x^{0}\in\Omega,\,\sigma\in(0,\operatorname{diam}\Omega)}\sigma^{-\mu}\int_{\Omega\cap B(x^{0},\sigma)}\|u(x)\|^{q}\,dx.\tag{2.7}$$

$\mathcal{L}^{q,\mu}(\Omega)$, $0\leqslant\mu\leqslant n+q$, $q\geqslant 1$, is the space of those functions $u\in L^{q}(\Omega)$ such that

$$[u]^{q}_{\mathcal{L}^{q,\mu}(\Omega)}=\sup_{x^{0}\in\Omega,\,\sigma\in(0,\operatorname{diam}\Omega)}\sigma^{-\mu}\int_{\Omega\cap B(x^{0},\sigma)}\|u(x)-u_{\Omega\cap B(x^{0},\sigma)}\|^{q}\,dx.\tag{2.8}$$

We say that $u\in H^{1,q,(\mu)}(\Omega)$, $0\leqslant\mu\leqslant n$, if $u\in H^{1,q}(\Omega)$ and $Du\in L^{q,\mu}(\Omega)$ and $H^{1,q,(\mu)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{H^{1,q,(\mu)}(\Omega)}=\|u\|_{L^{q,\mu}(\Omega)}+\|Du\|_{L^{q,\mu}(\Omega)}.\tag{2.9}$$

Moreover, if $Du\in L^{q,\mu}(\Omega)$, then $u\in\mathcal{L}^{q,\mu+q}(\Omega)$ (see [9, Proposition 3.7, p. 113]).

Important properties of these spaces are the following: $L^{p,n}(\Omega)=L^{\infty}(\Omega)$; if $0\leqslant\lambda<n$, then $\mathcal{L}^{p,\lambda}(\Omega)=L^{p,\lambda}(\Omega)$; if $n<\lambda\leqslant n+p$, then $\mathcal{L}^{p,\lambda}(\Omega)=C^{\gamma,0}(\bar{\Omega})$ with $\gamma=(\lambda-n)/p$, provided that Ω is sufficiently regular (for example, with the cone property) (see [2]).

We recall that, if $0\leqslant\mu<n$, then

$$G\in L^{q,\mu}(\Omega)\quad\Leftrightarrow\quad W(G)\in L^{2,\mu}(\Omega)$$

and the following inequality holds:

$$c\|G\|^{q}_{L^{q,\mu}(\Omega)}\leqslant\|W(G)\|^{2}_{L^{2,\mu}(\Omega)}\leqslant c\|G\|^{2}_{L^{q,\mu}(\Omega)}\big(1+\|G\|_{L^{q,\mu}(\Omega)}\big)^{q-2}.\tag{2.10}$$

In the sequel we need the following results.

Lemma 2.1. *There exists a positive constant* $c(q)$ *such that,* $\forall p, \bar{p} \in \mathbb{R}^k$,

$$\|W(p)\| + \|W(\bar{p})\| \leqslant 2W\big(\|p\| + \|\bar{p}\|\big) \leqslant c(q)\big\{\|W(p)\| + \|W(\bar{p})\|\big\}. \tag{2.11}$$

See [3, Lemma 2.I, p. 122].

Lemma 2.2. *If* $\mu > -1$, *there exist positive constants* $c(\mu)$ *and* $C(\mu)$ *such that, for every two vectors* a, b *in* $\mathbb{R}^N$, *we have*

$$c(\mu)\big(1 + \|a\| + \|b\|\big)^{\mu} \leqslant \int_0^1 \big(1 + \|a + tb\|\big)^{\mu}\, dt \leqslant C(\mu)\big(1 + \|a\| + \|b\|\big)^{\mu}. \tag{2.12}$$

See [3, Lemma 2.II, p.123].

Lemma 2.3. *Let* A *and* C *be bounded and open sets of* $\mathbb{R}^N$ *and* τ *be a mapping of class* C^1 *together with its inverse, from* $\bar{A}$ *into* $\bar{C}$. *Let* A^* *be an open set* $\subset\subset A$ *and* $C^* = \tau(A^*)$. *Then,* $\forall q > 1$ *and* $\forall \mu \in [0, n)$, *the mapping* $\varphi : u \to u \circ \tau$ *is a linear and continuous one together with its inverse, from* $L^{q,\mu}(C^*)$ *into* $L^{q,\mu}(A^*)$ *and from* $H^{1,q,(\mu)}(C^*)$ *into* $H^{1,q,(\mu)}(A^*)$.

See [3, Lemmas 2.IV and 2.V, p. 123].

Lemma 2.4. *Let* $\varphi(t)$ *and* $o(t)$ *be nonnegative functions defined in* $(0, d]$. *Suppose that* $\lim_{t\to 0} o(t) = 0$ *and* $\forall \sigma \in (0, d]$, $\forall t \in (0, 1)$

$$\varphi(t\sigma) \leqslant \big\{At^{\lambda} + o(\sigma)\big\}\varphi(\sigma) + K\sigma^{\mu}$$

with $0 < \mu < \lambda$, $A > 0$, $K \geqslant 0$. *Then for all* $\varepsilon < \lambda - \mu$ *there is a* $\sigma_\varepsilon \leqslant d$ *such that, if* $0 < \sigma \leqslant \sigma_\varepsilon$ *and* $t \in (0, 1)$,

$$\varphi(t\sigma) \leqslant (1 + A)t^{\lambda-\varepsilon}\varphi(\sigma) + KM(t\sigma)^{\mu},$$

where $M = M(A, \varepsilon, \lambda, \mu)$.

See [3, Lemma 2.VII, p. 125].

3. Global higher summability of the gradient

An essential tool for proving the global Hölder continuity of u is the global higher summability of the gradient.

First let us prove the following “Caccioppoli type” inequality:

Lemma 3.1. *Assume that conditions* (1.4), (1.5), (1.8), (1.9), (1.11), (1.12) *are fulfilled and* $g \in H^{1,q}(\Omega)$. *Let* $w \in H^{1,q}(\Omega)$ *be a solution of the strongly elliptic system*

$$\int_\Omega \sum_{i=1}^n \bigl(a^i(x, Dw + Dg)|D_i\varphi\bigr)\,dx = \int_\Omega \sum_{i=1}^n \bigl(F^i(x, w+g)|D_i\varphi\bigr)\,dx$$
$$+ \int_\Omega \bigl(F^0(x, w+g, Dw+Dg)|\varphi\bigr)\,dx, \quad \forall\varphi \in H_0^{1,q}(\Omega). \tag{3.1}$$

Then there exists a positive function $o(\sigma)$, which goes to zero with σ, such that for every couples of concentric balls $B(\sigma) \subset B(2\sigma) \subset \Omega$, it results

$$\int_{B(\sigma)} \bigl(1 + \|w\|^\delta + \|Dw\|\bigr)^q\,dx \leqslant c\sigma^{-q} \int_{B(2\sigma)} \|w - w_{B(2\sigma)}\|^q\,dx$$
$$+ c\sigma^n \Biggl(\fint_{B(2\sigma)} \bigl(1 + \|w\|^\delta + \|Dw\|\bigr)^{\frac{qn}{n+q}}\,dx \Biggr)^{\frac{n+q}{n}}$$
$$+ o(\sigma) \int_{B(2\sigma)} \bigl(1 + \|w\|^\delta + \|Dw\|\bigr)^q\,dx$$
$$+ c_1 \int_{B(2\sigma)} \bigl(1 + \|g\|^\delta + \|Dg\|\bigr)^q\,dx, \tag{3.2}$$

where $\delta = \max(\alpha, \beta/\gamma)$ and the constants c, c_1 do not depend on σ.

Proof. Having fixed $B(2\sigma) \subset \Omega$, let $\theta \in C_0^\infty(\mathbb{R}^n)$ be a function with the properties

$$0 \leqslant \theta \leqslant 1, \qquad \theta = 1 \quad \text{on } B(\sigma), \qquad \theta = 0 \quad \text{on } \mathbb{R}^n \setminus B(2\sigma),$$
$$\|D\theta\| \leqslant C\sigma^{-1},$$

with C a numerical constant. Let us assume in (3.1) $\varphi = \theta^q(w - w_{B(2\sigma)})$ and let us rewrite (3.1) in the following way:

$$\int_\Omega \sum_{i=1}^n \bigl(a^i(x, Dw)|\theta^q D_i w\bigr)\,dx$$
$$= \int_\Omega \sum_{i=1}^n \bigl(a^i(x, Dw) - a^i(x, Dw + Dg)|\theta^q D_i w\bigr)\,dx$$
$$- q \int_\Omega \sum_{i=1}^n \bigl(a^i(x, Dw + Dg)|\theta^{q-1} D_i\theta(w - w_{B(2\sigma)})\bigr)\,dx$$
$$+ \int_\Omega \sum_{i=1}^n \bigl(F^i(x, w+g)|\theta^q D_i w + q\theta^{q-1} D_i\theta(w - w_{B(2\sigma)})\bigr)\,dx$$
$$+ \int_\Omega \bigl(F^0(x, w+g, Dw+Dg)|\theta^q(w - w_{B(2\sigma)})\bigr)\,dx. \tag{3.3}$$

In virtue of the strong ellipticity condition (1.9), it results

$$\sum_{i=1}^{n}\bigl(a^i(x,Dw)|D_i w\bigr)=\sum_{i,j=1}^{n}\sum_{h,k=1}^{N}\left(\int_0^1 \frac{\partial a_h^i(x,tDw)}{\partial p_k^j}\,dt\right)D_j w D_i w$$
$$\geqslant \nu\|Dw\|^2\int_0^1\bigl(1+t\|Dw\|\bigr)^{q-2}\,dt \tag{3.4}$$

and taking into account Lemma 2.2 we get

$$\sum_{i=1}^{n}\bigl(a^i(x,Dw)|D_i w\bigr)\geqslant c(\nu,q)V^{q-2}(Dw)\|Dw\|^2. \tag{3.5}$$

Moreover, from condition (1.8) we derive

$$\sum_{i=1}^{n}\bigl(a^i(x,Dw)-a^i(x,Dw+Dg)|D_i w\bigr)$$
$$=-\sum_{i,j=1}^{n}\sum_{h,k=1}^{N}\left(\int_0^1 \frac{\partial a_h^i(x,Dw+tDg)}{\partial p_k^j}\,dt\right)D_j g D_i w$$
$$\leqslant M\int_0^1\bigl(1+\|Dw+tDg\|\bigr)^{q-2}\,dt\,\|Dg\|\|Dw\| \tag{3.6}$$

and by Lemma 2.2

$$\sum_{i=1}^{n}\bigl(a^i(x,Dw)-a^i(x,Dw+Dg)|D_i w\bigr)$$
$$\leqslant MV^{q-2}(Dw+Dg)\|Dg\|\|Dw\|$$
$$\leqslant c(M)V^{q-2}(Dw)\|Dg\|\|Dw\|+c(M)\|Dg\|^{q-1}\|Dw\|. \tag{3.7}$$

Then from (3.3) we obtain

$$c(\nu,q)\int_\Omega \theta^q V^{q-2}(Dw)\|Dw\|^2\,dx$$
$$\leqslant -q\int_\Omega\sum_{i=1}^{n}\bigl(a^i(x,Dw+Dg)|\theta^{q-1}D_i\theta(w-w_{B(2\sigma)})\bigr)\,dx$$
$$+\int_\Omega\sum_{i=1}^{n}\bigl(F^i(x,w+g)|\theta^q D_i w+q\theta^{q-1}D_i\theta(w-w_{B(2\sigma)})\bigr)\,dx$$
$$+\int_\Omega\bigl(F^0(x,w+g,Dw+Dg)|\theta^q(w-w_{B(2\sigma)})\bigr)\,dx$$

$$+c(M)\int_\Omega V^{q-2}(Dw)\theta^q\|Dg\|\|Dw\|\,dx+c(M)\int_\Omega \theta^q\|Dg\|^{q-1}\|Dw\|\,dx$$
$$\leqslant |A|+|B|+|C|+|D|+|E|.$$

Let us estimate the terms of the right hand side.

For what concerns $|A|$, taking into account the condition (1.10) and applying the Hölder inequality, we get

$$\begin{aligned}|A| &\leqslant c\int_\Omega \theta^{q-1}\big(1+\|Dw\|+\|Dg\|\big)^{q-1}\sigma^{-1}\|w-w_{B(2\sigma)}\|\,dx\\ &\leqslant \varepsilon\int_\Omega \theta^q\big(1+\|Dw\|\big)^q\,dx+c\sigma^{-q}\int_{B(2\sigma)}\|w-w_{B(2\sigma)}\|^q\,dx\\ &\quad+c\int_{B(2\sigma)}\|Dg\|^q\,dx, \end{aligned}\tag{3.8}$$

where the constants depend on q and M.

Similarly, $\forall\varepsilon>0$ and taking into account that $\alpha q\leqslant\delta q$,

$$\begin{aligned}|B| &\leqslant c\int_\Omega \theta^{q-1}\big(1+\|w\|^\alpha+\|g\|^\alpha\big)^{q-1}\theta\|Dw\|\,dx\\ &\quad+c\int_\Omega \theta^{q-1}\big(1+\|w\|^\alpha+\|g\|^\alpha\big)^{q-1}\sigma^{-1}\|w-w_{B(2\sigma)}\|\,dx\\ &\leqslant \varepsilon\int_\Omega \theta^q\|Dw\|^q\,dx+c(\varepsilon)\int_\Omega \theta^q\big(1+\|w\|^{\delta q}\big)\,dx\\ &\quad+c\sigma^{-q}\int_{B(2\sigma)}\|w-w_{B(2\sigma)}\|^q\,dx+c(\varepsilon)\int_\Omega \theta^q\big(1+\|g\|^{\delta q}\big)\,dx. \end{aligned}$$

Moreover, being $\delta q\geqslant n/(n-1)$,

$$\begin{aligned}\int_\Omega \theta^q\|w\|^{\delta q}\,dx &\leqslant c\int_{B(2\sigma)}\|w-w_{B(2\sigma)}\|^{\delta q}\,dx+c\sigma^n\|w_{B(2\sigma)}\|^{\delta q}\\ &\leqslant \bigg(\int_{B(2\sigma)}\|Dw\|^{\frac{\delta qn}{n+\delta q}}\,dx\bigg)^{\frac{n+\delta q}{n}}+c\sigma^n\bigg(\fint_{B(2\sigma)}\|w\|^{\frac{\delta qn}{n+q}}\,dx\bigg)^{\frac{n+q}{n}}\\ &\leqslant c\sigma^{n-\delta(n-q)}\bigg(\int_{B(2\sigma)}\|Dw\|^q\,dx\bigg)^{\delta}+c\sigma^n\bigg(\fint_{B(2\sigma)}\|w\|^{\frac{\delta qn}{n+q}}\,dx\bigg)^{\frac{n+q}{n}}. \end{aligned}$$

Because we have

$$\int_{\Omega} \theta^q \, dx \leqslant \int_{B(2\sigma)} 1 \, dx \leqslant c\sigma^n \left(\fint_{B(2\sigma)} dx \right)^{\rho}, \quad \forall \rho > 0,$$

we get, for each $\varepsilon > 0$,

$$\begin{aligned} |B| \leqslant{} & \varepsilon \int_{\Omega} \theta^q \|Dw\|^q \, dx + c\sigma^n \left(\fint_{B(2\sigma)} \left(1 + \|w\|^{\delta}\right)^{\frac{qn}{n+q}} dx \right)^{\frac{n+q}{n}} \\ & + o_1(\sigma) \int_{B(2\sigma)} \|Dw\|^q \, dx + c\sigma^{-q} \int_{B(2\sigma)} \|w - w_{B(2\sigma)}\|^q \, dx \\ & + c \int_{B(2\sigma)} \left(1 + \|g\|^{\delta q}\right) dx, \end{aligned} \tag{3.9}$$

where

$$o_1(\sigma) = c\sigma^{n-\delta(n-q)} \left(\int_{B(2\sigma)} \|Dw\|^q \, dx \right)^{\delta - 1}$$

and the constants c depend on q, M, C_1.

$$\begin{aligned} |C| \leqslant{} & c \int_{\Omega} \theta^q \left(1 + \|w\|^{\beta} + \|g\|^{\beta} + \|Dw\|^{\gamma} + \|Dg\|^{\gamma}\right)^{q-1} \|w - w_{B(2\sigma)}\| \, dx \\ \leqslant{} & c \int_{\Omega} \theta^q \left(1 + \|w\|^{\beta} + \|Dw\|^{\gamma}\right)^{q-1} \|w - w_{B(2\sigma)}\| \, dx \\ & + c \int_{\Omega} \theta^q \left(1 + \|g\|^{\beta} + \|Dg\|^{\gamma}\right)^{q-1} \|w - w_{B(2\sigma)}\| \, dx \\ \leqslant{} & c \left(\int_{B(2\sigma)} \|w - w_{B(2\sigma)}\|^{\frac{q}{q-\gamma(q-1)}} dx \right)^{1 - \frac{\gamma(q-1)}{q}} \\ & \times \left[\left(\int_{B(2\sigma)} \left(1 + \|w\|^{\frac{\beta q}{\gamma}} + \|Dw\|^q\right) dx \right)^{\frac{\gamma(q-1)}{q}} \right. \\ & \left. + \left(\int_{B(2\sigma)} \left(1 + \|g\|^{\frac{\beta q}{\gamma}} + \|Dg\|^q\right) dx \right)^{\frac{\gamma(q-1)}{q}} \right]. \end{aligned}$$

Since (see [4, (2-24)])

$$\left(\int_{B(2\sigma)} \|w - w_{B(2\sigma)}\|^{\frac{q}{q-\gamma(q-1)}}\,dx\right)^{\frac{q-\gamma(q-1)}{q}} \leqslant \sigma^{1-\frac{n(\gamma-1)}{q'}}\left(\int_{B(2\sigma)} \|Dw\|^q\,dx\right)^{\frac{1}{q}},$$

we get

$$\begin{aligned}|C| \leqslant c\sigma^{1-\frac{n(\gamma-1)}{q'}}\Bigg[&\left(\int_{B(2\sigma)} \big(1+\|w\|^{\delta q}+\|Dw\|^q\big)\,dx\right)^{1+\frac{\gamma-1}{q'}}\\ &+\left(\int_{B(2\sigma)} \big(1+\|g\|^{\delta q}+\|Dg\|^q\big)\,dx\right)^{1+\frac{\gamma-1}{q'}}\Bigg]\end{aligned}$$

and hence

$$\begin{aligned}|C| \leqslant{}& co_2(\sigma)\int_{B(2\sigma)} \big(1+\|w\|^{\delta q}+\|Dw\|^q\big)\,dx\\ &+ c_1\int_{B(2\sigma)} \big(1+\|g\|^{\delta q}+\|Dg\|^q\big)\,dx,\end{aligned} \tag{3.10}$$

where

$$o_2(\sigma) = c\sigma^{1-\frac{n(\gamma-1)}{q'}}\left(\int_{B(2\sigma)} \big(1+\|w\|^{\delta q}+\|Dw\|^q\big)\,dx\right)^{\frac{\gamma-1}{q'}},$$

$$c_1 = c\left(\int_{\Omega} \big(1+\|g\|^{\delta q}+\|Dg\|^q\big)\,dx\right)^{\frac{\gamma-1}{q'}}$$

and the constants depend on q, M, C_0.

For what concerns D and E we have

$$\begin{aligned}D &\leqslant \int_{\Omega} \theta^{q-1}\big(1+\|Dw\|\big)^{q-1}\theta\|Dg\|\,dx\\ &\leqslant \varepsilon\int_{\Omega} \theta^q\big(1+\|Dw\|\big)^q\,dx + c(\varepsilon)\int_{B(2\sigma)} \|Dg\|^q\,dx\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}E &\leqslant \int_{\Omega} \theta^{q-1}\|Dg\|^{q-1}\theta\|Dw\|\,dx\\ &\leqslant \varepsilon\int_{\Omega} \theta^q\|Dw\|^q\,dx + c(\varepsilon)\int_{B(2\sigma)} \|Dg\|^q\,dx.\end{aligned} \tag{3.12}$$

From (3.8)–(3.12) we get, for each $\varepsilon > 0$,

$$\begin{aligned}
&\int_{\Omega} \theta^q V^{q-2}(Dw)\|Dw\|^2\,dx \\
&\quad\leqslant c\sigma^{-q}\int_{B(2\sigma)}\|w-w_{B(2\sigma)}\|^q\,dx + c\sigma^n\left(\fint_{B(2\sigma)}\left(1+\|w\|^{\delta}\right)^{\frac{qn}{n+q}}dx\right)^{\frac{n+q}{n}} \\
&\qquad + o(\sigma)\int_{B(2\sigma)}\left(1+\|w\|^{\delta q}+\|Dw\|^q\right)dx + c_1\int_{B(2\sigma)}\left(1+\|g\|^{\delta q}+\|Dg\|^q\right)dx \\
&\qquad + \varepsilon\int_{\Omega}\theta^q\left(1+\|w\|^{\delta}+\|Dw\|\right)^q dx \\
&\quad= \mathcal{M} + c_1\int_{B(2\sigma)}\left(1+\|g\|^{\delta q}+\|Dg\|^q\right)dx \\
&\qquad + \varepsilon\int_{\Omega}\theta^q\left(1+\|w\|^{\delta}+\|Dw\|\right)^q dx,
\end{aligned}\tag{3.13}$$

where

$$o(\sigma) = o_1(\sigma) + o_2(\sigma)$$

goes to zero with σ in virtue of the absolute continuity of integral. Moreover, $c = c(\nu, q, M, C_1, C_0)$, $c_1 = c_1(\nu, q, M, C_1, C_0, \|g\|_{H^{1,q}(\Omega)})$ do not depend on σ.

From (3.13) we get

$$\begin{aligned}
\int_{\Omega}\theta^q\|Dw\|^q\,dx &\leqslant \mathcal{M} + c_1\int_{B(2\sigma)}\left(1+\|g\|^{\delta q}+\|Dg\|^q\right)dx \\
&\qquad + \varepsilon\int_{\Omega}\theta^q\left(1+\|w\|^{\delta}+\|Dw\|\right)^q dx,
\end{aligned}$$

and being

$$\int_{\Omega}\theta^q\left(1+\|w\|^{\delta q}\right)dx \leqslant c\mathcal{M},\tag{3.14}$$

from (3.13) and (3.14) we get, for ε sufficiently small,

$$\begin{aligned}
\int_{B(\sigma)}\left(1+\|w\|^{\delta}+\|Dw\|\right)^q dx &\leqslant c\sigma^{-q}\int_{B(2\sigma)}\|w-w_{B(2\sigma)}\|^q\,dx \\
&\quad + c\sigma^n\left(\fint_{B(2\sigma)}\left(1+\|w\|^{\delta}+\|Dw\|\right)^{\frac{qn}{n+q}}dx\right)^{\frac{n+q}{n}}
\end{aligned}$$

$$+\,o(\sigma) \int\limits_{B(2\sigma)} \left(1 + \|w\|^{\delta} + \|Dw\|\right)^q \, dx + c_1 \int\limits_{B(2\sigma)} \left(1 + \|g\|^{\delta} + \|Dg\|\right)^q \, dx. \tag{3.15}$$

Hence the assert. □

Lemma 3.2. *Assume that conditions* (1.4), (1.5), (1.8), (1.9), (1.11), (1.12) *are fulfilled and* $g \in H^{1,q^*}(\Omega)$. *If* $w \in H^{1,q}(\Omega)$ *is a solution of the strongly elliptic system*

$$\int\limits_{\Omega} \sum_{i=1}^{n} \left(a^i(x, Dw + Dg) | D_i\varphi\right) dx = \int\limits_{\Omega} \sum_{i=1}^{n} \left(F^i(x, w + g) | D_i\varphi\right) dx$$
$$+ \int\limits_{\Omega} \left(F^0(x, w + g, Dw + Dg) | \varphi\right) dx, \quad \forall \varphi \in H_0^{1,q}(\Omega), \tag{3.16}$$

then there exists a number $1 < \tilde{r} < q^*/q$ *and a constant* $\sigma_0(w) > 0$ *such that* $Du \in L^{q\tilde{r}}_{\mathrm{loc}}(\Omega)$ *and* $\forall B(2\sigma) \subset \Omega$ *with* $\sigma < \sigma_0$ *it results*

$$\left(\fint\limits_{B(\sigma)} \left(1 + \|w\|^{\delta} + \|Dw\|\right)^{q\tilde{r}} dx \right)^{\frac{1}{\tilde{r}}} \leqslant K \fint\limits_{B(2\sigma)} \left(1 + \|w\|^{\delta} + \|Dw\|\right)^q dx$$
$$+ K \left(\fint\limits_{B(2\sigma)} \left(1 + \|g\|^{\delta} + \|Dg\|\right)^{q\tilde{r}} dx \right)^{\frac{1}{\tilde{r}}}, \tag{3.17}$$

where $\delta = \max(\alpha, \beta/\gamma)$ *and the constant* K *does not depend on* σ.

Proof. By Poincaré inequality it follows

$$\sigma^{-q} \int\limits_{B(2\sigma)} \|w - w_{B(2\sigma)}\|^q \, dx \leqslant c\sigma^n \left(\fint\limits_{B(2\sigma)} \|Dw\|^{\frac{nq}{n+q}} \, dx \right)^{\frac{n+q}{n}}.$$

Hence, if we set

$$U = \left(1 + \|w\|^{\delta} + \|Dw\|\right)^{\frac{qn}{n+q}},$$
$$G = \left(1 + \|g\|^{\delta} + \|Dg\|\right)^{\frac{qn}{n+q}},$$

from Caccioppoli's inequality (3.2) it follows

$$\fint\limits_{B(\sigma)} U^{\frac{n+q}{n}} \, dx \leqslant c \left(\fint\limits_{B(2\sigma)} U \, dx \right)^{\frac{n+q}{n}} + o(\sigma) \fint\limits_{B(2\sigma)} U^{\frac{n+q}{n}} \, dx + c \fint\limits_{B(2\sigma)} G^{\frac{n+q}{n}} \, dx.$$

Then in virtue of Ghering–Giaquinta–Modica lemma (see, for instance, [2, p. 125]) and of the fact that $o(\sigma)$ goes to zero with σ, the assert follows. □

Lemma 3.3. *Assume that conditions* (1.4), (1.5), (1.8), (1.9), (1.11), (1.12) *are fulfilled and* $g \in H^{1,q^*}(B^+(1))$. *If* $w \in H^{1,q}(B^+(1))$ *is a solution of the strongly elliptic problem*

$$\begin{cases} \int_{B^+(1)} \sum_{i=1}^n (a^i(x, w+g, Dw+Dg)|D_i\varphi)\,dx \\ \quad = \int_{B^+(1)} \sum_{i=1}^n (F^i(x, w+g)|D_i\varphi)\,dx \\ \qquad + \int_{B^+(1)} (F^0(x, w+g, Dw+Dg)|\varphi)\,dx, \quad \forall \varphi \in H_0^{1,q}(B^+(1)), \\ w(x) = 0 \quad \text{on } \Gamma, \end{cases} \tag{3.18}$$

then there exists a constant $\bar\sigma_0(w) > 0$ *and a number* $1 < r' < q^*/q$ *such that* $Dw \in L^{qr'}_{\mathrm{loc}}(B^+(1))$ *and* $\forall B^+(2\sigma) \subset B^+(1)$ *with* $\sigma < \bar\sigma_0$ *it results*

$$\begin{aligned} \left(\fint_{B^+(\sigma)} \big(1 + \|w\|^\delta + \|Dw\|\big)^{qr'}\,dx \right) &\leqslant K \fint_{B^+(2\sigma)} \big(1 + \|w\|^\delta + \|Dw\|^q\big)\,dx \\ &\quad + K \left(\fint_{B^+(2\sigma)} \big(1 + \|g\|^\delta + \|Dg\|\big)^{qr'}\,dx \right)^{\frac{1}{r'}}, \end{aligned} \tag{3.19}$$

where $\delta = \max(\alpha, \beta/\gamma)$ *and* K *is a positive constant which does not depend on* σ.

Proof. Let us choose $\sigma < 1/2$ and a function $\theta \in C_0^\infty(\mathbb{R}^n)$ having the following properties:

$$\begin{aligned} &0 \leqslant \theta \leqslant 1, \qquad \theta = 1 \quad \text{in } B(\sigma), \qquad \theta = 0 \quad \text{in } \mathbb{R}^n \setminus B(2\sigma), \\ &\|D\theta\| \leqslant C\sigma^{-1}, \end{aligned} \tag{3.20}$$

with C a numerical constant. Taking into account that $w = 0$ on Γ, in (3.18) we can assume $\varphi = \theta^q w$ and, arguing as in the proof of Lemma 3.1, we get the "Caccioppoli type" estimate

$$\begin{aligned} &\int_{B^+(\sigma)} \big(1 + \|w\|^\delta + \|Dw\|\big)^q\,dx \\ &\quad \leqslant c\sigma^{-q} \int_{B^+(2\sigma)} \|w\|^q\,dx + c\sigma^n \left(\fint_{B^+(2\sigma)} \big(1 + \|w\|^\delta + \|Dw\|\big)^{\frac{qn}{n+q}}\,dx \right)^{\frac{n+q}{n}} \\ &\qquad + o(\sigma) \int_{B^+(2\sigma)} \big(1 + \|w\|^\delta + \|Dw\|\big)^q\,dx \\ &\qquad + c_1 \int_{B^+(2\sigma)} \big(1 + \|g\|^\delta + \|Dg\|\big)^q\,dx. \end{aligned} \tag{3.21}$$

Now, taking into account that

$$w(x) = 0 \quad \text{on } \Gamma$$

we can apply the Poincaré inequality

$$\sigma^{-q} \int\limits_{B^+(2\sigma)} \|w\|^q \, dx \leqslant c\sigma^n \left(\fint\limits_{B^+(2\sigma)} \|Dw\|^{\frac{nq}{n+q}} \, dx \right)^{\frac{n+q}{n}}$$

and hence, to obtain (3.19), we can repeat the same argument of Lemma 3.2. □

Lemma 3.4. *Let the conditions* (1.4), (1.5), (1.8), (1.9), (1.11), (1.12) *be fulfilled, let* $\partial\Omega$ *be of class* C^2 *and* $g \in H^{1,q^*}(\Omega)$. *If* $w \in H_0^{1,q}(\Omega)$ *is a solution to the Dirichlet problem*

$$\begin{cases} \sum_{i=1}^n D_i a^i(x, Dw + Dg) = \sum_{i=1}^n D_i F^i(x, w + g) - F^0(x, w + g, Dw + Dg), \\ w = 0 \quad \text{on } \partial\Omega, \end{cases}$$

then there exists $r > 1$ *such that* $Dw \in L^{qr}(\Omega)$.

Proof. It is enough to use the usual covering procedure (see [3, Lemmas 2.V, 2.VI, 2.VII and Section 8]). □

From Lemma 3.4 we immediately derive the global Hölder continuity of u for $q = n$. For this reason in the sequel we shall confine ourselves to the case $q < n$.

4. Interior local regularity result

For what follows it is useful to recall some important results due to Campanato.

Theorem 4.1. *If* $u \in H^{1,q}(\Omega)$, $q \geqslant 2$, *is a solution of the basic system*

$$\sum_i D_i a^i(Du) = 0 \quad \text{in } \Omega, \tag{4.1}$$

under conditions (1.5), (1.8), (1.9), *then, for every ball* $B(\sigma) = B(x^0, \sigma) \subset\subset \Omega$ *and* $\forall t \in (0, 1)$,

$$\int\limits_{B(t\sigma)} \|W(Du)\|^2 \, dx \leqslant ct^\lambda \int\limits_{B(\sigma)} \|W(Du)\|^2 \, dx, \tag{4.2}$$

where

$$\lambda = \min\{2 + \varepsilon, n\},$$

the constant c *does not depend on* t, σ, x^0 *and* $\varepsilon = \varepsilon(\nu, M, \nu)$ *is a suitable number* $0 < \varepsilon < 1$. *See* [3, *Theorem* 3.1, *p.* 128].

Now we can present the main result on interior local regularity. Let $u \in H^{1,q}(\Omega)$ be a solution of the Dirichlet problem

$$\begin{cases} u-g\in H_0^{1,q}(\Omega), \\ \sum_i D_i a^i(x,Du)=\sum_i D_i F^i - F^0 \quad \text{in } \Omega. \end{cases}$$

Set

$$w=u-g.$$

The vector $w\in H^{1,q}(\Omega)$ is a solution of the system

$$\sum_i D_i a^i(x,Dw+Dg) = \sum_i D_i F^i(x,w+g) - F^0(x,w+g,Dw+Dg) \quad \text{in } \Omega \tag{4.3}$$

in the following sense:

$$\int_\Omega \sum_i \bigl(a^i(x,Dw+Dg)|D_i\varphi\bigr)\,dx = \int_\Omega \sum_i (F^i|D_i\varphi)\,dx + \int_\Omega (F^0|\varphi)\,dx, \quad \forall\varphi\in H_0^{1,q}(\Omega). \tag{4.4}$$

We have the following result:

Theorem 4.2. *Assume that conditions* (1.4), (1.5), (1.8), (1.9), (1.11) *and* (1.12) *are fulfilled. If*

$$g\in H^{1,\frac{nq}{n-q},(\frac{\mu}{q-1})}(\Omega), \quad 0<\frac{\mu}{q-1}<\lambda, \tag{4.5}$$

and $w\in H^{1,q}(\Omega)$ is a solution of the system

$$\sum_i D_i a^i(x,Dw+Dg)=\sum_i D_i F^i(x,w+g)-F^0(x,w+g,Dw+Dg) \quad \text{in } \Omega,$$

then for every $\Omega^*\subset\subset\Omega$ we have

$$Dw\in L^{q,\frac{\mu}{q-1}}(\Omega^*). \tag{4.6}$$

Proof. Let Ω^* be an open set: $\Omega^*\subset\subset\Omega$ and $d=\operatorname{dist}(\bar\Omega^*,\partial\Omega)$. Fix the ball $B(\sigma)=B(x^0,\sigma)$ with $x^0\in\Omega^*$ and $\sigma\leqslant d$.

In $B(\sigma)$ we decompose w as $v-z$, where z is a solution of the Dirichlet problem

$$\begin{cases} z\in H_0^{1,q}(\Omega), \\ \sum_i D_i a^i(x^0,Dz+Dw+Dg) \\ \quad =\sum_i D_i[a^i(x,Dw+Dg)-F^i]+F^0 \quad \text{in } B(\sigma), \end{cases} \tag{4.7}$$

while $v\in H^{1,q}(B(\sigma))$ is a solution of the system

$$\sum_i D_i a^i(x^0,Dv+Dg)=0 \quad \text{in } B(\sigma). \tag{4.8}$$

(4.7) means that $\forall\varphi\in H_0^{1,q}(B(\sigma))$

$$\int\limits_{B(\sigma)} \sum_i \bigl(a^i(x^0, Dz + Dw + Dg) - a^i(x^0, Dw + Dg) | D_i\varphi\bigr)\,dx$$
$$= \int\limits_{B(\sigma)} \sum_i \bigl(a^i(x, Dw + Dg) - a^i(x^0, Dw + Dg) | D_i\varphi\bigr)\,dx$$
$$- \int\limits_{B(\sigma)} \sum_i (F^i | D_i\varphi)\,dx - \int\limits_{B(\sigma)} (F^0 | \varphi)\,dx. \tag{4.9}$$

Assuming $\varphi = z$, setting

$$B_{ij} = \int\limits_0^1 A_{ij}(x, Dw + Dg + tDz)\,dt,$$

taking into account ellipticity condition (1.9) and Lemma 2.2, we obtain

$$c \int\limits_{B(\sigma)} \bigl(1 + \|Dw\| + \|Dg\| + \|Dz\|\bigr)^{q-2} \|Dz\|^2\,dx$$
$$\leqslant \int\limits_{B(\sigma)} \sum_i \bigl\|a^i(x, Dw + Dg) - a^i(x^0, Dw + Dg)\bigr\| \|Dz\|\,dx$$
$$+ \int\limits_{B(\sigma)} \sum_i \|F^i\| \|D_i z\|\,dx + \int\limits_{B(\sigma)} \|F^0\| \|z\|\,dx$$
$$= |A| + |B| + |C|. \tag{4.10}$$

On the other hand, by hypothesis (1.4) we have $\forall \varepsilon > 0$

$$|A| = \int\limits_{B(\sigma)} \sum_i \bigl\|a^i(x, Dw + Dg) - a^i(x^0, Dw + Dg)\bigr\| \|Dz\|\,dx$$
$$\leqslant c\omega(\sigma) \int\limits_{B(\sigma)} V^{q-2}(Dw + Dg) \|Dw + Dg\| \|Dz\|\,dx$$
$$\leqslant \varepsilon \int\limits_{B(\sigma)} V^{q-2}(Dw + Dg) \|Dz\|^2\,dx$$
$$+ c\omega^2(\sigma) \int\limits_{B(\sigma)} \|W(Dw + Dg)\|^2\,dx. \tag{4.11}$$

Now, by means of conditions (1.11), (1.12), (4.5) and by Lemma 3.4, we have

$$\int\limits_{B(\sigma)} \|Dz\|\,dx \leqslant \varepsilon \int\limits_{B(\sigma)} \|Dz\|^q\,dx + c\sigma^n, \tag{4.12}$$

$$\begin{aligned}
\int_{B(\sigma)} \|Dz\|\|w\|^{\alpha(q-1)}\,dx &\leqslant \left(\int_{B(\sigma)} \|Dz\|^q\,dx\right)^{\frac{1}{q}}\left(\int_{B(\sigma)} \|w\|^{\alpha q}\,dx\right)^{\frac{1}{q'}}\\
&\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\int_{B(\sigma)} \|w\|^{q^*}\,dx + c\sigma^n\\
&\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\sigma^{\frac{r-1}{r}\frac{n^2}{n-q}}\|w\|^{\frac{nq}{n-q}}_{H^{1,qr}(B(\sigma))} + c\sigma^n, \qquad (4.13)
\end{aligned}$$

and

$$\begin{aligned}
\int_{B(\sigma)} \|Dz\|\|g\|^{\alpha(q-1)}\,dx &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\int_{B(\sigma)} \|g\|^{q^*}\,dx + c\sigma^n\\
&\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\sigma^{\frac{\mu}{q-1}}\|g\|^{q^*}_{L^{q^*,\mu/(q-1)}(B(\sigma))} + c\sigma^n. \qquad (4.14)
\end{aligned}$$

Therefore by (4.12), (4.13), (4.14)

$$\begin{aligned}
|B| &= \int_{B(\sigma)} \sum_i \|F^i\|\|D_i z\|\,dx\\
&\leqslant \varepsilon \int_{B(\sigma)} \|W(Dz)\|^2\,dx\\
&\quad + c\Big(\sigma^n + \sigma^{\frac{r-1}{r}\frac{n^2}{n-q}}\|w\|^{\frac{nq}{n-q}}_{H^{1,qr}(B(\sigma))} + \sigma^{\frac{\mu}{q-1}}\|g\|^{q^*}_{L^{q^*,\mu/(q-1)}(B(\sigma))}\Big). \qquad (4.15)
\end{aligned}$$

In the same way, taking into account the following estimates

$$\int_{B(\sigma)} \|z\|\,dx \leqslant \left(\int_{B(\sigma)} \|z\|^{q^*}\,dx\right)^{\frac{1}{q^*}}(\sigma^n)^{\frac{1}{q''}} \leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\sigma^{\frac{n(q-1)+q}{q-1}}, \qquad (4.16)$$

$$\begin{aligned}
\int_{B(\sigma)} \|z\|\|w\|^{\beta(q-1)}\,dx &\leqslant \left(\int_{B(\sigma)} \|z\|^{q^*}\,dx\right)^{\frac{1}{q^*}}\left(\int_{B(\sigma)} \|w\|^{\beta(q-1)q''}\,dx\right)^{\frac{1}{q''}}\\
&\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\|w\|^{\frac{q^*q'}{q''}}_{H^{1,q}(B(\sigma))} + c\sigma^{\frac{n(q-1)+q}{q-1}}\\
&\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\sigma^{n(\frac{r-1}{r})\frac{n(q-1)+q}{(n-q)(q-1)}}\|w\|^{q\frac{n(q-1)+q}{(n-q)(q-1)}}_{H^{1,qr}(B(\sigma))} + c\sigma^{\frac{n(q-1)+q}{q-1}}, \qquad (4.17)
\end{aligned}$$

$$\int_{B(\sigma)} \|z\|\|g\|^{\beta(q-1)}\,dx \leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\left(\int_{B(\sigma)} \|g\|^{q^*}\,dx\right)^{\frac{q'}{q''}} + c\sigma^{\frac{n(q-1)+q}{q-1}}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\sigma^{\frac{\mu}{(q-1)^2}\frac{n(q-1)+q}{n}} \|g\|_{L^{q^*,\mu/(q-1)}(B(\sigma))}^{\frac{q[n(q-1)+q]}{(n-q)(q-1)}} + c\sigma^{\frac{n(q-1)+q}{q-1}}, \tag{4.18}$$

$$\int_{B(\sigma)} \|z\|\|Dw\|^{\gamma(q-1)}\,dx$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\left(\int_{B(\sigma)} \|Dw\|^q\,dx\right)^{\frac{q'}{q''}} + c\sigma^{\frac{n(q-1)+q}{q-1}}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\sigma^{n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})} \|Dw\|_{L^{qr}(B(\sigma))}^{q(1+\frac{q}{n(q-1)})} + c\sigma^{\frac{n(q-1)+q}{q-1}}, \tag{4.19}$$

$$\int_{B(\sigma)} \|z\|\|Dg\|^{\gamma(q-1)}\,dx \leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\left(\int_{B(\sigma)} \|Dg\|^q\,dx\right)^{\frac{q'}{q''}} + c\sigma^{\frac{n(q-1)+q}{q-1}}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,dx + c\sigma^{\frac{\mu}{q-1}\frac{n-q}{n}\frac{n(q-1)+q}{n(q-1)} + \frac{q}{q-1}\frac{n(q-1)+q}{n}} \|Dg\|_{L^{q^*,\mu/(q-1)}(B(\sigma))}^{\frac{qq'}{q''}}$$

$$+ c\sigma^{\frac{n(q-1)+q}{q-1}}, \tag{4.20}$$

we have

$$|C| = \int_{B(\sigma)} \|F^0\|\|z\|\,dx$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|W(Dz)\|^2\,dx + c\Big(\sigma^{\frac{n(q-1)+q}{q-1}} + \sigma^{n(\frac{r-1}{r})\frac{n(q-1)+q}{(n-q)(q-1)}} \|w\|_{H^{1,qr}(B(\sigma))}^{q\frac{n(q-1)+q}{(n-q)(q-1)}}$$

$$+ \sigma^{\frac{\mu}{(q-1)^2}\frac{n(q-1)+q}{n}} \|g\|_{L^{q^*,\mu/(q-1)}(B(\sigma))}^{\frac{q[n(q-1)+q]}{(n-q)(q-1)}} + \sigma^{n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})} \|Dw\|_{L^{qr}(B(\sigma))}^{q(1+\frac{q}{n(q-1)})}$$

$$+ \sigma^{\frac{\mu}{q-1}\frac{n-q}{n}\frac{n(q-1)+q}{n(q-1)} + \frac{q}{q-1}\frac{n(q-1)+q}{n}} \|Dg\|_{L^{q^*,\mu/(q-1)}(B(\sigma))}^{\frac{qq'}{q''}}\Big). \tag{4.21}$$

From (4.11), (4.15), (4.21) it follows

$$
\begin{aligned}
& c \int\limits_{B(\sigma)} \bigl(1+\|Dw\|+\|Dg\|+\|Dz\|\bigr)^{q-2}\|Dz\|^2\,dx \\
&\quad \leqslant \varepsilon \int\limits_{B(\sigma)} V^{q-2}(Dw+Dg)\|Dz\|^2\,dx + \varepsilon \int\limits_{B(\sigma)} \|W(Dz)\|^2\,dx \\
&\qquad + c\Biggl\{\omega^2(\sigma)\int\limits_{B(\sigma)} \|W(Dw+Dg)\|^2\,dx + \sigma^n + \sigma^{\frac{n(q-1)+q}{q-1}} \\
&\qquad + \sigma^{\frac{r-1}{r}\frac{n^2}{n-q}}\|w\|_{H^{1,qr}(B(\sigma))}^{\frac{nq}{n-q}} + \sigma^{\frac{\mu}{q-1}}\|g\|_{L^{q^*,\mu/(q-1)}(B(\sigma))}^{q^*} \\
&\qquad + \sigma^{n(\frac{r-1}{r})\frac{n(q-1)+q}{(n-q)(q-1)}}\|w\|_{H^{1,qr}(B(\sigma))}^{q\frac{n(q-1)+q}{(n-q)(q-1)}} + \sigma^{\frac{\mu}{(q-1)^2}\frac{n(q-1)+q}{n}}\|g\|_{L^{q^*,\mu/(q-1)}(B(\sigma))}^{\frac{q[n(q-1)+q]}{(n-q)(q-1)}} \\
&\qquad + \sigma^{n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})}\|Dw\|_{L^{qr}(B(\sigma))}^{q(1+\frac{q}{n(q-1)})} \\
&\qquad + \sigma^{\frac{\mu}{q-1}\frac{n-q}{n}\frac{n(q-1)+q}{n(q-1)}+\frac{q}{q-1}\frac{n(q-1)+q}{n}}\|Dg\|_{L^{q^*,\mu/(q-1)}(B(\sigma))}^{\frac{qq'}{q''}}\Biggr\},
\end{aligned}
\tag{4.22}
$$

where the constant depend on $\varepsilon, q, \nu, C_0, C_1, d$. Then for a suitable choose of ε we obtain

$$
\begin{aligned}
& \int\limits_{B(\sigma)} \|W(Dz)\|^2\,dx \leqslant c\Biggl\{\omega^2(\sigma)\int\limits_{B(\sigma)} \|W(Dw+Dg)\|^2\,dx + \sigma^n \\
&\quad + \sigma^{\frac{r-1}{r}\frac{n^2}{n-q}}\|w\|_{H^{1,qr}(B(\sigma))}^{\frac{nq}{n-q}} + \sigma^{\frac{\mu}{q-1}}\|g\|_{L^{q^*,\mu/(q-1)}(B(\sigma))}^{q^*} + \sigma^{\frac{n(q-1)+q}{q-1}} \\
&\quad + \sigma^{n(\frac{r-1}{r})\frac{n(q-1)+q}{(n-q)(q-1)}}\|w\|_{H^{1,qr}(B(\sigma))}^{q\frac{n(q-1)+q}{(n-q)(q-1)}} + \sigma^{\frac{\mu}{(q-1)^2}\frac{n(q-1)+q}{n}}\|g\|_{L^{q^*,\mu/(q-1)}(B(\sigma))}^{\frac{q[n(q-1)+q]}{(n-q)(q-1)}} \\
&\quad + \sigma^{n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})}\|Dw\|_{L^{qr}(B(\sigma))}^{q(1+\frac{q}{n(q-1)})} \\
&\quad + \sigma^{\frac{\mu}{q-1}\frac{n-q}{n}\frac{n(q-1)+q}{n(q-1)}+\frac{q}{q-1}\frac{n(q-1)+q}{n}}\|Dg\|_{L^{q^*,\mu/(q-1)}(B(\sigma))}^{\frac{qq'}{q''}}\Biggr\}.
\end{aligned}
\tag{4.23}
$$

The vector $(v+g)$ is a solution of the basic system (4.8), and then fundamental estimate (4.2) holds: $\forall t \in (0,1)$

$$
\int\limits_{B(t\sigma)} \|W(Dv+Dg)\|^2\,dx \leqslant ct^{\lambda}\int\limits_{B(\sigma)} \|W(Dv+Dg)\|^2\,dx.
$$

Hence (see Lemma 2.1)

$$
\int\limits_{B(t\sigma)} \|W(Dv)\|^2\,dx \leqslant ct^{\lambda}\int\limits_{B(\sigma)} \|W(Dv)\|^2\,dx + c\int\limits_{B(\sigma)} \|W(Dg)\|^2\,dx. \tag{4.24}
$$

As $w = v - z$ in $B(\sigma)$, it follows from (4.23), (4.24) that $\forall t \in (0,1)$

$$\int_{B(t\sigma)} \|W(Dw)\|^2\,dx \leqslant c \int_{B(t\sigma)} \|W(Dv)\|^2 + \|W(Dz)\|^2\,dx$$
$$\leqslant ct^\lambda \int_{B(\sigma)} \|W(Dw)\|^2\,dx + c\int_{B(\sigma)} \|W(Dg)\|^2\,dx + c\int_{B(\sigma)} \|W(Dz)\|^2\,dx$$
$$\leqslant c\big(t^\lambda + \omega^2(\sigma)\big) \int_{B(\sigma)} \|W(Dw)\|^2\,dx + c\sigma^{\min\{\frac{\mu}{q-1}, n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})\}}\mathcal{M}, \tag{4.25}$$

where

$$\mathcal{M} = 1 + \|W(Dg)\|^2_{L^{2,\mu/(q-1)}(\Omega)} + \|w\|^{\frac{nq}{n-q}}_{H^{1,qr}(B(\sigma))} + \|g\|^{q^*}_{L^{q^*,\mu/(q-1)}(\Omega)} + \|w\|^{q\frac{n(q-1)+q}{(n-q)(q-1)}}_{H^{1,qr}(B(\sigma))} + \|g\|^{\frac{q[n(q-1)+q]}{(n-q)(q-1)}}_{L^{q^*,\mu/(q-1)}(\Omega)}$$
$$+ \|Dw\|^{q(1+\frac{q}{n(q-1)})}_{L^{qr}(\Omega)} + \|Dg\|^{\frac{qq'}{q''}}_{L^{q^*,\mu/(q-1)}(\Omega)}. \tag{4.26}$$

By Lemma 2.4 and from (4.25) it follows that

$$\forall \varepsilon < \lambda - \min\left\{\frac{\mu}{q-1}, n\left(\frac{r-1}{r}\right)\left(1 + \frac{q}{n(q-1)}\right)\right\}$$

$\exists \sigma_\varepsilon \leqslant d$ such that $\forall \sigma < \sigma_\varepsilon$ and $\forall t \in (0,1)$

$$\int_{B(t\sigma)} \|W(Dw)\|^2\,dx \leqslant (1+c)t^{\min\{\frac{\mu}{q-1}, n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})\}} \int_{B(\sigma)} \|W(Dw)\|^2\,dx$$
$$+ c\mathcal{M}(t\sigma)^{\min\{\frac{\mu}{q-1}, n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})\}}, \tag{4.27}$$

where the constant c does not depend on $\mathcal{M}$. This implies that $\forall \sigma < \sigma_\varepsilon$

$$\int_{B(\sigma)} \|W(Dw)\|^2\,dx \leqslant c\sigma^{\min\{\frac{\mu}{q-1}, n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})\}}$$
$$\times \left\{\sigma_\varepsilon^{-\min\{\frac{\mu}{q-1}, n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})\}} \int_\Omega \|W(Dw)\|^2\,dx + \mathcal{M}\right\} \tag{4.28}$$

and consequently

$$\|W(Dw)\|^2_{L^{2,\min\{\frac{\mu}{q-1}, n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})\}}(\Omega^*)} \leqslant c\left\{\int_\Omega \|W(Dw)\|^2\,dx + \mathcal{M}\right\}. \tag{4.29}$$

Now, if

$$\frac{\mu}{q-1} \leqslant n\left(\frac{r-1}{r}\right)\left(1 + \frac{q}{n(q-1)}\right),$$

the theorem is proved. Otherwise, if

$$n\left(\frac{r-1}{r}\right)\left(1 + \frac{q}{n(q-1)}\right) < \frac{\mu}{q-1},$$

from (4.29) it follows

$$Dw \in L^{q,n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})}(\Omega^*). \tag{4.30}$$

From (4.30), repeating the same arguments as above and using a suitable choose of the set Ω^*, we have that $\forall t \in (0,1)$

$$\int_{B(t\sigma)} \|W(Dw)\|^2\,dx \leqslant c\big(t^\lambda + \omega^2(\sigma)\big) \int_{B(\sigma)} \|W(Dw)\|^2\,dx + c\sigma^{\min\{\frac{\mu}{q-1}, n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})^2\}}\mathcal{M}.$$

Now, if

$$\frac{\mu}{q-1} \leqslant n\left(\frac{r-1}{r}\right)\left(1+\frac{q}{n(q-1)}\right)^2,$$

the theorem is proved.

Otherwise, if

$$n\left(\frac{r-1}{r}\right)\left(1+\frac{q}{n(q-1)}\right)^2 < \frac{\mu}{q-1},$$

repeating the same arguments as above a finite number of times, we find an integer n' such that

$$n\left(\frac{r-1}{r}\right)\left(1+\frac{q}{n(q-1)}\right)^{n'} > n$$

and

$$\int_{B(t\sigma)} \|W(Dw)\|^2\,dx \leqslant c\big(t^\lambda + \omega^2(\sigma)\big) \int_{B(\sigma)} \|W(Dw)\|^2\,dx + c\sigma^{\min\{\frac{\mu}{q-1}, n(\frac{r-1}{r})(1+\frac{q}{n(q-1)})^{n'}\}}\mathcal{M}. \tag{4.31}$$

Because $n \geqslant \lambda > \mu/(q-1)$ the assert of Theorem 4.2 is completely proved. $\square$

5. Regularity near the boundary

Let $a^i(x,p)$ be vector of $\mathbb{R}^N$, defined in $\Lambda^+ = B^+(1) \times \mathbb{R}^{nN}$, of class C^1 in p and $F^0(x,u,p)$, $F^i(x,u)$, $i = 1,2,\dots,n$, be vectors of $\mathbb{R}^N$ defined, respectively, in $B^+(1) \times \mathbb{R}^N \times \mathbb{R}^{nN}$ and $B^+(1) \times \mathbb{R}^N$, measurable in x, continuous in u and p.

Theorem 5.1. *If $u \in H^{1,q}(B^+(1))$ is a solution of the problem*

$$\begin{cases} \sum_i D_i a^i(Du) = 0 & \text{in } B^+(1), \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{5.1}$$

under hypotheses (1.5), (1.8), (1.9), *where* Ω *is replaced by* $B^+(1)$, *then* $\forall\sigma \leqslant 1$ *and* $\forall t \in (0,1)$

$$\int_{B^+(t\sigma)} \|W(Du)\|^2\,dx \leqslant ct^\lambda \int_{B^+(\sigma)} \|W(Du)\|^2\,dx,$$

where the constant c *does not depend on* t, σ *and* $\lambda = \min(2+\varepsilon, n)$ (*with* $\varepsilon \neq n-2$).

See [3, Theorem 6.II, p. 141].

Theorem 5.2. *Assume that conditions* (1.4), (1.5), (1.8), (1.9), (1.11) *and* (1.12)[3] *are fulfilled. If*

$$g \in H^{1,\frac{nq}{n-q},(\frac{\mu}{q-1})}\big(B^+(1)\big), \quad 0 < \frac{\mu}{q-1} < \lambda,$$

and $w \in H^{1,q}(B^+(1))$ *is a solution of the problem*

$$\begin{cases} w = 0 \quad \text{on } \Gamma, \\ \sum_i D_i a^i(x, Dw + Dg) = \sum_i D_i F^i(x,u) - F^0(x,u,Du) \quad \text{in } B^+(1), \end{cases} \tag{5.2}$$

then, for every $R < 1$, *we get* $Dw \in L^{q,\mu/(q-1)}(B^+(R))$.

The proof of this theorem requests to get estimates similar to the ones obtained in the interior case (see Theorem 4.2). The only difference is that we must use Theorem 5.1 instead of Theorem 4.1. The details of a similar proof can be found in [10, Theorem 4.2].

Now Theorem 1.1 can be obtained from the interior local regularity theorem (Theorem 4.2) together with the boundary local regularity theorem (Theorem 5.2) and using local coordinate charts. The details of such a procedure can be found in [10, Section 5]. In this way, taking into account that $u = w + g$, with $Dg \in L^{nq/(n-q),\mu/(q-1)}(\Omega)$, we get $Du \in L^{q,\mu/(q-1)}(\Omega)$. Then $u \in \mathcal{L}^{q,\mu/(q-1)+q}(\Omega)$ (see Section 2) and, if $n - q < \mu/(q-1) < \lambda$, also $u \in C^{0,\alpha}(\bar{\Omega})$ with $\alpha = 1 - n/q + \mu/(q-1)(1/q)$ (see Section 2).

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